UNP JOURNAL MANAGEMENT CENTER

math.ppj.unp.ac.id
p-ISSN 2716-0726
e–ISSN 2716-0734

Indexed

An Analysis of Cardano's Formula to Solved Cubic Equation

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Article History
Vol. 4, No. 2, 2025

Subject Areas:
Mathematics

Keywords:
Cubic equation, Canonic Equation, Cardano' Formula, Root Characteristics

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DOI:
https://doi.org/10.24036/rmj.v4i2.64

How to Cite:
Darajat, R.A., & Helma. (2025). An Analysis of Cardano's Formula to Solved Cubic Equation. *Rangkiang Mathematics Journal* 4(2), 101-110. https://doi.org/10.24036/rmj.v4i2.64

Abstract- Cubic equation in complex is equation with the form $az^3 + bz^2 + cz + d = 0,\ a,b,c,d \in R;\ a \neq 0, z \in C$. The equation has solution that is often referred to as the root of the equation. In the method commonly is used to find the root of the equation is usually obtained the real number, but for complex root it can't be determined. In determined the complex root is used Cardano's formula. This formula is reduced the form of the cubic equation to canonical equation form, namely $z^3 + pz + q = 0$. This research a basic research. In this research data was collected from various sources in the form of related theories. Beginning with analyzing the graph and it characteristics, tracing the Cardano's formula and analyzing it proof. Furthermore, is formed a formula with the characteristics of the roots owned by a cubic equation. The research results show that Cardano's formula which has root for $z = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}$ can be determined of the roots with values of $p = -\frac{b^2}{3a^2} + \frac{c}{a}$ and $q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}$. The characteristics of the roots of cubic equation $z^3 + pz + q = 0$ are depended by the values of p, q and Δ, where $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$. Thus, eight characteristics of the roots of cubic equation can be obtained based on the values of p, q and Δ.

1. Introduction

In mathematical calculations, a function is often found. The function is written as $f(x)$, shows the value given by f to x (Mánik, 2021). Cubic function in the form,

$$f(x) = ax^3 + bx^2 + cx + d, \quad a,b,c,d \in \boldsymbol{R};\ a \neq 0, x \in \boldsymbol{R}$$

This function is a function in real numbers. For cubic function in complex can be expressed as,

$$f(z) = az^3 + bz^2 + cz + d, \quad a,b,c,d \in \boldsymbol{R};\ a \neq 0, z \in \boldsymbol{C}$$

This function can be written as an equation, which has the form,

$$az^3 + bz^2 + cz + d = 0, \qquad a, b, c, d \in \boldsymbol{R};\ a \neq 0, z \in \boldsymbol{C}$$

This equation has a solution. The solution is often referred to as the root of the equation. The existence of these roots is guaranteed by the theorem, "Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$ with degree $n \geq 1$ and $a_n \neq 0$ has at least one root. From this it follows that $P(z) = 0$ has exactly n roots, due attention being paid to multiplicities of roots" (Spigel, 1987). The cubic holds a double fascination since not only is it interesting in its own right, but its solution is also the key to solving quartics (Nickalls, 2009).

There are two methods used in determining the solution of the equation, namely numerical method and analytical method. Numerical method produce values in the form of approximations (Susila, 1993), while the analytical or exact method is a solution method that gives true or actual results (Vulandari, 2021). One of the commonly used analytical methods is Horner's method (Nurtaniyahya et al., 2023). Horner's method is a method for approximating the real roots of an alqebraic equation (Coolidge & Gray, 2023). In determining the root, the option that results in the equation equating to zero is chosen. This method can't be used to determine the roots of complex numbers. Cubic equation can also be determined by the factoring process (Musli, 2018). The factors of a given algebraic expression consist of two or more algebraic expressions which when multiplied together produce the given expression (Spigel, 1987). These methods, however, have certain limitations. Therefore, a method is needed to find solutions for the complex roots of cubic equation $az^3 + bz^2 + cz + d = 0,\ a, b, c, d \in \boldsymbol{R};\ a \neq 0, z \in \boldsymbol{C}$. A world-renowned mathematician from Pavia, Italy, namely Girolamo Cardano. in 1545, Cardano published the *Artis magnae sive de regulis algebraicis (Ars Magna)* which contain solution to the problems of such equation (Rowe et al., 2014). Cardano gives the general solution of reduced cubic polynomials that is, cubic equation without second degree terms (Lestari et al., 2020). In addition to being used to find general solutions to cubic equations, the Cardano method can also be used to determine the discriminant and characteristics of the roots of cubic equations (Janson, 2010).

Research on cubic equation has attracted the interest of researchers (Helma & Amalita, 2003; Nurtaniyahya et al., 2023; Okereke et al., 2014). The results of research conducted by Helma & Amalita (2003) show that the way to find the characteristics of a cubic equation that have real roots. The results of research conducted by Okereke, Okereke et al. (2014) show that the way to find the solution of cubic equations by using the concept of quadratic factoring. Furthemore, the results of research conducted by Nurtaniyahya, Nurtaniyahya et al. (2023) discuss the method of determining roots by linking the roots of cubic equations based on on the coefficient of the Cardano's formula. Based on this explanation, then this research is discussed in a structured manner regarding graph analysis and its correlation with the forming of canonical equation, the forming of Cardano's formula, and Cardano's formula with root characteristics.

2. Methods

This type of research is basic research. Basic research aims to develop theories and doesn't focus on direct practical applications Sugiyono (2016). In this research data was collected from various sources in the form of theories related to the research, so that a solution to the existing problem can be obtained. The steps taken in this research are as follows:

1. Analyzing the graph and its characteristics.
   In this section, the graph will be analyzed, and changes under certain conditions will be observed. By examining the changes in the graph, its characteristics can be analyzed.
2. Tracing Cardano's formula and analyzing its proof.
   In this section, the canonical equation is first formed. From the canonical equation, the Cardano's formula will be formed.
3. Forming a formula based on the characteristics of the roots of a cubic equation.

In this section, after exploring the Cardano's formula, the effect of coefficients and delta (Δ) of the canonical equation $z^3 + pz + q = 0,\ p, q \in \boldsymbol{R}$ will be analyzed.

3. Results and Discussion

(a) Graphical Analysis

Consider the following cubic equation:

$$3x^3 + 8\,x^2 + 2\,x + 2 = 0$$

Cubic function can be represented by a graph as shown Figure 1.

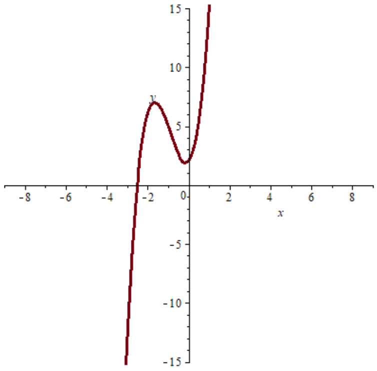


Figure 1. Graph of $f(x) = 3x^3 + 8\,x^2 + 2\,x + 2$

By setting $f(x) = 0$ then the cubic function becomes the cubic equation $3x^3 + 8\,x^2 + 2\,x + 2 = 0$. Let $x = z - t$, then

$$3(z-t)^3 + 8\,(z-t)^2 + 2\,(z-t) + 2 = 0$$
$$z^3 + \left(-3t + \frac{8}{3}\right)z^2 + \left(3t^2 - \frac{16}{3}t + \frac{2}{3}\right)z + \left(-t^3 + \frac{8}{3}t^2 - \frac{2}{3}t + \frac{2}{3}\right) = 0$$

If $t = \frac{8}{9}$, then

$$z^3 + \left(-3\left(\frac{8}{9}\right) + \frac{8}{3}\right)z^2 + \left(3\left(\frac{8}{9}\right)^2 - \frac{16}{3}\left(\frac{8}{9}\right) + \frac{2}{3}\right)z + \left(-\left(\frac{8}{9}\right)^3 + \frac{8}{3}\left(\frac{8}{9}\right)^2 - \frac{2}{3}\left(\frac{8}{9}\right) + \frac{2}{3}\right) = 0$$
$$z^3 - \frac{46}{27}z + \frac{1.078}{729} = 0$$

It can be seen that the equation $3x^3 + 8\,x^2 + 2\,x + 2 = 0$ can be reduced to $z^3 + pz + q = 0$. From the new equation, the graph in Figure 2 is obtained

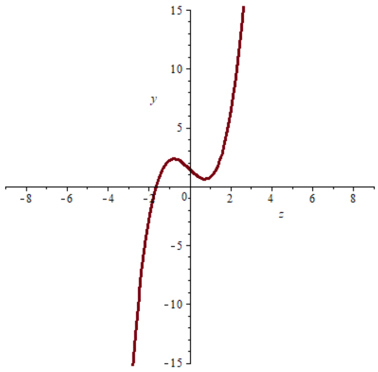


Figure 2. Graph of $f(z) = z^3 - \frac{46}{27}z + \frac{1.078}{729}$

Then, it will be analyzed in four forms:

1. If the equation $3x^3 + 8\,x^2 + 2\,x + 2 = 0$ is changed to,

$$-3x^3 + 8\,x^2 + 2\,x + 2 = 0$$

By assuming $x = z - t$, then

$$-3(z-t)^3 + 8\,(z-t)^2 + 2\,(z-t) + 2 = 0$$
$$z^3 + \left(-3t - \frac{8}{3}\right)z^2 + \left(3t^2 + \frac{16}{3}t - \frac{2}{3}\right)z + \left(-t^3 - \frac{8}{3}t^2 + \frac{2}{3}t - \frac{2}{3}\right) = 0$$

If $t = -\frac{8}{9}$, then

$$z^3 + \left(-3\left(-\frac{8}{9}\right) - \frac{8}{3}\right)z^2 + \left(3\left(-\frac{8}{9}\right)^2 + \frac{16}{3}\left(-\frac{8}{9}\right) - \frac{2}{3}\right)z + \left(-\left(-\frac{8}{9}\right)^3 - \frac{8}{3}\left(-\frac{8}{9}\right)^2 + \frac{2}{3}\left(-\frac{8}{9}\right) - \frac{2}{3}\right) = 0$$
$$z^3 - \frac{82}{27}z - \frac{1.942}{729} = 0$$

It can be seen that the equation $3x^3 + 8\,x^2 + 2\,x + 2 = 0$ can be reduced to $z^3 + pz + q = 0$.

2. If the equation $3x^3 + 8\,x^2 + 2\,x + 2 = 0$ is changed to,

$$3x^3 - 8\,x^2 + 2\,x + 2 = 0$$

By assuming $x = z - t$, then

$$3(z-t)^3 - 8\,(z-t)^2 + 2\,(z-t) + 2 = 0$$

math.ppj.unp.ac.id Rangkiang Mathematics Journal Vol 4, No. 2, 2025

$$z^3+\left(-3t-\frac{8}{3}\right)z^2+\left(3t^2+\frac{16}{3}t+\frac{2}{3}\right)z+\left(-t^3-\frac{8}{3}t^2-\frac{2}{3}t+\frac{2}{3}\right)=0$$

If $t=-\frac{8}{9}$, then

$$z^3+\left(-3\left(-\frac{8}{9}\right)-\frac{8}{3}\right)z^2+\left(3\left(-\frac{8}{9}\right)^2+\frac{16}{3}\left(-\frac{8}{9}\right)+\frac{2}{3}\right)z+\left(-\left(-\frac{8}{9}\right)^3-\frac{8}{3}\left(-\frac{8}{9}\right)^2-\frac{2}{3}\left(-\frac{8}{9}\right)+\frac{2}{3}\right)=0$$

$$z^3-\frac{46}{27}z-\frac{106}{729}=0$$

It can be seen that the equation $3x^3+8x^2+2x+2=0$ can be reduced to $z^3+pz+q=0$.

3. If the equation $3x^3+8x^2+2x+2=0$ is changed to,

$$3x^3+8x^2-2x+2=0$$

By assuming $x=z-t$, then

$$3(z-t)^3+8(z-t)^2-2(z-t)+2=0$$

$$z^3+\left(-3t+\frac{8}{3}\right)z^2+\left(3t^2-\frac{16}{3}t-\frac{2}{3}\right)z+\left(-t^3+\frac{8}{3}t^2+\frac{2}{3}t+\frac{2}{3}\right)=0$$

If $t=\frac{8}{9}$, then

$$z^3+\left(-3\left(\frac{8}{9}\right)+\frac{8}{3}\right)z^2+\left(3\left(\frac{8}{9}\right)^2-\frac{16}{3}\left(\frac{8}{9}\right)-\frac{2}{3}\right)z+\left(-\left(\frac{8}{9}\right)^3+\frac{8}{3}\left(\frac{8}{9}\right)^2+\frac{2}{3}\left(\frac{8}{9}\right)+\frac{2}{3}\right)=0$$

$$z^3-\frac{82}{27}z+\frac{1.942}{729}=0$$

It can be seen that the equation $3x^3+8x^2+2x+2=0$ can be reduced to $z^3+pz+q=0$.

4. If the equation $3x^3+8x^2+2x+2=0$ is changed to,

$$3x^3+8x^2+2x-2=0$$

$$3(z-t)^3+8(z-t)^2+2(z-t)-2=0$$

$$z^3+\left(-3t+\frac{8}{3}\right)z^2+\left(3t^2-\frac{16}{3}t+\frac{2}{3}\right)z+\left(-t^3+\frac{8}{3}t^2-\frac{2}{3}t-\frac{2}{3}\right)=0$$

If $t=\frac{8}{9}$, then

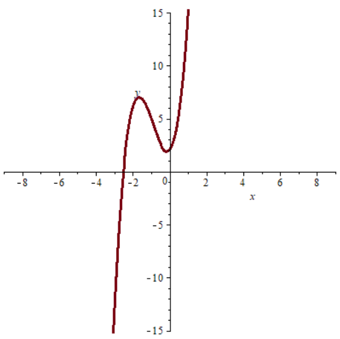
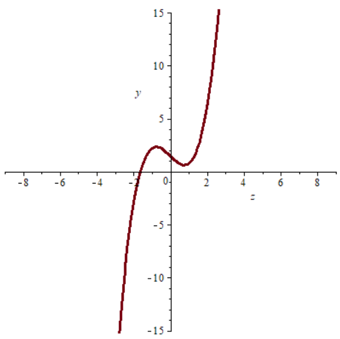
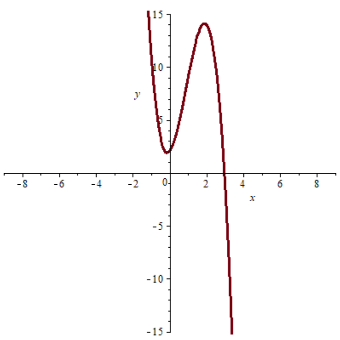
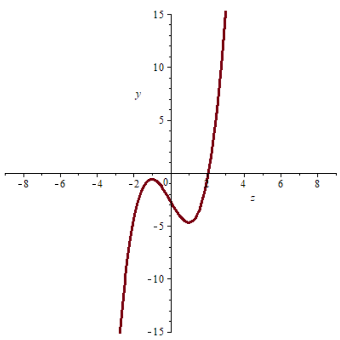
$$z^3+\left(-3\left(\frac{8}{9}\right)+\frac{8}{3}\right)z^2+\left(3\left(\frac{8}{9}\right)^2-\frac{16}{3}\left(\frac{8}{9}\right)+\frac{2}{3}\right)z+\left(-\left(\frac{8}{9}\right)^3+\frac{8}{3}\left(\frac{8}{9}\right)^2-\frac{2}{3}\left(\frac{8}{9}\right)-\frac{2}{3}\right)=0$$

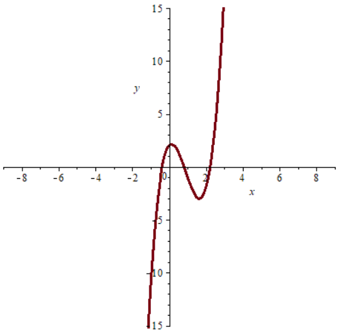
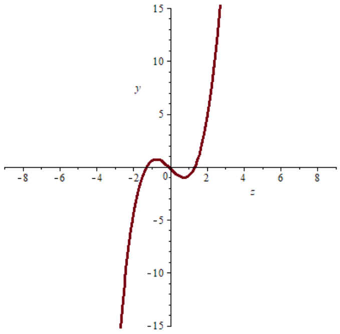
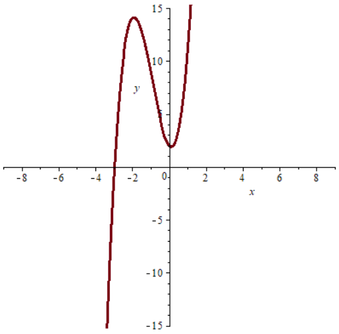
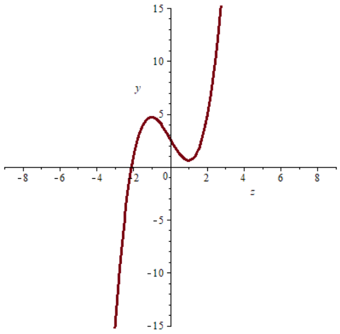
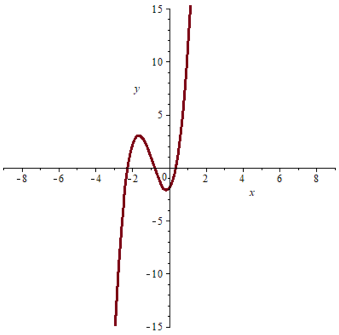
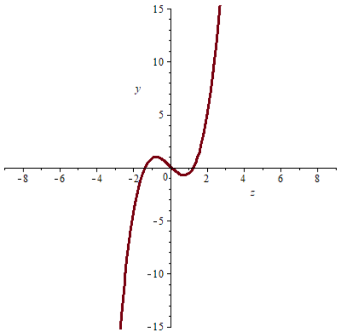
$$z^3-\frac{46}{27}z+\frac{106}{729}=0$$

It can be seen that the equation $3x^3+8x^2+2x+2=0$ can be reduced to $z^3+pz+q=0$.

The comparison table of the graphs as a whole can be seen in Table 1

Table 1. Comparison of Graph

Form	Graph 1	Graph 2	Description
Basic	 $f(x)=3x^3+8x^2+2x+2$	 $f(z)=z^3-\frac{46}{27}z+\frac{1.078}{729}$	Let's assume $x=z-t$ for $3x^3+8x^2+2x+2=0$, then the equation shifts its root by $t=\frac{8}{9}$.
1			Let's assume $x=z-t$ for $-3x^3+8x^2+2x+2=0$, then the equation shifts its root by $t=-\frac{8}{9}$.

	$f(x) = -3x^3 + 8\,x^2 + 2\,x + 2$	$f(z) = z^3 - \frac{82}{27}z - \frac{1.942}{729}$	
2	 $f(x) = 3x^3 - 8\,x^2 + 2\,x + 2$	 $f(z) = z^3 - \frac{46}{27}z - \frac{106}{729}$	Let's assume $x = z - t$ for $3x^3 - 8\,x^2 + 2\,x + 2 = 0$, then the equation shifts its root by $t = -\frac{8}{9}$.
3	 $f(x) = 3x^3 + 8\,x^2 - 2\,x + 2$	 $f(z) = z^3 - \frac{82}{27}z + \frac{1.942}{729}$	Let's assume $x = z - t$ for $3x^3 + 8\,x^2 - 2\,x + 2 = 0$, then the equation shifts its root by $t = \frac{8}{9}$.
4	 $f(x) = 3x^3 + 8\,x^2 + 2\,x - 2$	 $f(z) = z^3 - \frac{46}{27}z + \frac{106}{729}$	Let's assume $x = z - t$ for $3x^3 + 8\,x^2 + 2\,x - 2 = 0$, then the equation shifts its root by $t = \frac{8}{9}$.

Based on the analysis of the graphs, it can be obsserved that the graph pattern shows a shift in the root by t. Every cubic equation can be reduced to the simple form which is the form $z^3 + pz + q = 0$. This equation is referred to as the cnical equation, this equation serves as the fundamental form for deriving the Cardano' formula. Subsequently, the canonical equation will be proved.

(b) Determine the Canonical Equation

Examining the cubic equation $ax^3 + bx^2 + cx + d = 0,\ a, b, c, d \in \boldsymbol{R};\ a \neq 0$. Let $x = z - t$. Then,

$a(z-t)^3 + b(z-t)^2 + c(z-t) + d = 0$

$z^3 + \left(-3t + \frac{b}{a}\right)z^2 + \left(3t^2 - \frac{2b}{a}t + \frac{c}{a}\right)z + \left(-t^3 + \frac{b}{a}t^2 - \frac{c}{a}t + \frac{d}{a}\right) = 0$

If $t = \frac{b}{3a}$, then

$$z^3 + \left(-3\left(\frac{b}{3a}\right) + \frac{b}{a}\right)z^2 + \left(3\left(\frac{b}{3a}\right)^2 - \frac{2b}{a}\left(\frac{b}{3a}\right) + \frac{c}{a}\right)z + \left(-\left(\frac{b}{3a}\right)^3 + \frac{b}{a}\left(\frac{b}{3a}\right)^2 - \frac{c}{a}\left(\frac{b}{3a}\right) + \frac{d}{a}\right) = 0$$

math.ppj.unp.ac.id Rangkiang Mathematics Journal Vol 4, No. 2, 2025

$z^3 + \left(-\frac{b^2}{3a^2} + \frac{c}{a}\right) z + \left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right) = 0$

The equation $ax^3 + bx^2 + cx + d = 0$ can be reduced to $z^3 + pz + q = 0$, with the values

$$p = -\frac{b^2}{3a^2} + \frac{c}{a}$$

and

$$q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}$$

The equation $z^3 + pz + q = 0$ is the simpler cubic equation, also known as the canonical equation.

(c) Cardano's Formula

From the canonical equation, Cardano's formula will be formed.
Let $z = w - t$. Then $z^3 + pz + q = 0$ becomes,
$(w - t)^3 + p(w - t) + q = 0$
$w^3 + (w - t)(-3wt + p) + (-t^3) + q = 0$
If $(w - t)(-3wt + p) = 0$, then $w = t$ or $t = \frac{p}{3w}$, $w \neq 0$
For $w = t$, it follows that $z = 0$ which isn't possible.
For $t = \frac{p}{3w}$, where $w \neq 0$, it follows that $z = w - \frac{p}{3w}$
Thus,

$$w^3 + \left(-\left(\frac{p}{3w}\right)^3\right) + q = 0$$

Multiplying of the equation by $\frac{w^3}{w^3}$, the following result is obtained

$$\frac{(w^3)^2 + q(w^3) - \frac{p^3}{27}}{w^3} = 0$$

or

$$(w^3)^2 + q(w^3) - \frac{p^3}{27} = 0$$

By using the quadratic formula (abc), the following result is obtained

$$w_{1,2}^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Then, let us assume

$$w_1^3 = U = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

and

$$w_2^3 = V = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

There are three solutions from U and three solutions from V.
Let u_1, u_2, u_3 be the roots of U and v_1, v_2, v_3 be the roots of V.
Consider the product of U and V,

$$U \cdot V = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right) = -\frac{p^3}{27}$$

Since $U \cdot V = -\frac{p^3}{27}$ it follows that $u_j = -\frac{p}{3v_j}$, $j = 1, 2, 3$. If $w = u_j$, $j = 1, 2, 3$. Therefore, from the previous equation,

$$z = u_j + v_j, \qquad j = 1,2,3$$

Thus,

$$z_{1,2,3} = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3}$$

This is a Cardano's formula.

(d) Cardano's Formula with Root Characteristics

In this section, the roots will be determined by examining the characteristics of the roots of cubic equation $z^3 + pz + q = 0,\ p, q \in \boldsymbol{R}$. Consider the cubic equation,

$$z^3 + pz + q = 0, \qquad p, q \in \boldsymbol{R}$$

Using the Cardano' formula, the roots of the equation above are:

$$z_{1,2,3} = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3}$$

A root can be determined to be either a real or complex number with considering the form $\frac{q^2}{4} + \frac{p^3}{27}$. The form is denoted by delta (Δ). The formula above can expressed as:

$$z_{1,2,3} = \left(-\frac{q}{2} + \sqrt{\Delta}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{\Delta}\right)^{1/3}$$

In order to determine the value of Δ, formerly observed the coefficients of the equation that is values of p and q.

The values of p, q and Δ will be analyzed as follows:

1. Case 1:

If $p \neq 0, p > 0$ and $q \neq 0$.

$$z^3 + pz + q = 0$$

By assuming one of the roots as $z_1 = u + v$, it is obtained that

$(u + v)^3 + p(u + v) + q = 0$

$u^3 + 3u^2v + 3uv^2 + v^3 + p(u + v) + q = 0$

$(u^3 + v^3 + q) + (u + v)(3uv + p) = 0$

The values of u and v are determined, with $u^3 + v^3 + q = 0$ and $3uv + p = 0$ are fixed. Thus,

$u^3 + v^3 = -q,$

and

$uv = -p$

$\Leftrightarrow uv = -\frac{p}{3}$

$\Leftrightarrow (uv)^3 = \left(-\frac{p}{3}\right)^3$

$\Leftrightarrow u^3v^3 = -\left(\frac{p}{3}\right)^3$

If u^3 and v^3 are the roots of a quadratic equation. With the equation,

$$s^2 + qs - \left(\frac{p}{3}\right)^3 = 0$$

Then,

$u^3 = \frac{-q+\sqrt{q^2-4\left(-\left(\frac{p}{3}\right)^3\right)}}{2}$

$\Leftrightarrow u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$

and

$v^3 = \frac{-q-\sqrt{q^2-4\left(-\left(\frac{p}{3}\right)^3\right)}}{2}$

$\Leftrightarrow v = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$

Substitute the values of u and v into z_1=u+v which is one of the roots,

$$z_1 = u + v$$

$$z_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

For the other roots, which are z_2 and z_3.

$(z - z_1)(z - z_2)(z - z_3) = 0$

$z^3 - (z_1 + z_2 + z_3)z^2 + (z_1z_2 + z_2z_3 + z_1z_3)z - (z_1z_2z_3) = 0$

With,

$z_1 + z_2 + z_3 = 0$

$z_1z_2 + z_2z_3 + z_1z_3 = p$

$z_1 z_2 z_3 = -q$
Then,
$z_1 + z_2 + z_3 = 0$
$z_2 + z_3 = -z_1 = -(u+v)$
$z_1 z_2 z_3 = -q$
$z_2 z_3 = \frac{-q}{z_1} = \frac{u^3+v^3}{u+v} = \frac{(u+v)(u^2-uv+v^2)}{u+v} = u^2 - uv + v^2$
Consider that $z_2 + z_3$ and $z_2 z_3$ are the roots of the quadratic equation,

$$z^2 - \left(-(u+v)\right)z + (u^2 - uv + v^2) = 0$$
$$z^2 + (u+v)z + (u^2 - uv + v^2) = 0$$

By using the quadratic formula (abc), the following result is obtained

$$z_2 = \frac{-(u+v)+\sqrt{(u+v)^2-4(u^2-uv+v^2)}}{2}$$
$$= -\frac{1}{2}(u+v) + \frac{1}{2}i\sqrt{3}(u-v)$$
$$z_3 = \frac{-(u+v)-\sqrt{(u+v)^2-4(u^2-uv+v^2)}}{2}$$
$$= -\frac{1}{2}(u+v) - \frac{1}{2}i\sqrt{3}(u-v)$$

Substitute the respective values of z_1, z_2, and z_3 into the value of $x = z - t$, where $t = \frac{b}{3a}$.

$$x_1 = u + v - \frac{b}{3a}$$
$$x_2 = -\frac{1}{2}(u+v) + \frac{1}{2}i\sqrt{3}(u-v) - \frac{b}{3a}$$
$$x_3 = -\frac{1}{2}(u+v) - \frac{1}{2}i\sqrt{3}(u-v) - \frac{b}{3a}$$

To be more specific, it is analyzed based on the following conditions:

a. Case 1(a)

If $p > 0$, then $\Delta > 0$. Based on the case 1, the value of x_1 is a real number and x_2, x_3 are complex number. Thus, the characteristics of the roots are one real root and two conjugate complex roots.

b. Case 1(b)

If $p < 0$, then:

1) $\Delta > 0$.

Since $\Delta > 0$, should be $\frac{q^2}{4} > \left|\frac{p^3}{27}\right|$. Based on the case 1, the value of x_1 is a real number and x_2, x_3 are complex number. Thus, the characteristics of the roots are one real root and two conjugate complex roots.

2) $\Delta < 0$

Since $\Delta < 0$, should be $\frac{q^2}{4} < \left|\frac{p^3}{27}\right|$. Based on the case 1, the value of x_1, x_2, and x_3 are real number. Thus, the characteristics of the roots are three distinct real roots.

3) $\Delta = 0$

Since $\Delta = 0$, should be $\frac{q^2}{4} = \left|\frac{p^3}{27}\right|$. Based on the case 1, the value of x_1, x_2, and x_3 are real number. Thus, the characteristics of the roots are three real roots with two equal real roots.

2. Case 2:

If $p \neq 0$ and $q = 0$. The equation $z^3 + pz + q = 0$, becomes
$z^3 + pz = 0$
$z(z^2 + p) = 0$
$z_1 = 0$, and $z_{2,3}^2 + p = 0 \Leftrightarrow z_{2,3} = \pm\sqrt{-p}$
Thus, $z_1 = 0$, $z_2 = \sqrt{-p}$ dan $z_3 = -\sqrt{-p}$.
Substitute z_1, z_2 and z_3 to $x = z - t$, with $t = \frac{b}{3a}$.

$$x_1 = -\frac{b}{3a}$$
$$x_2 = \sqrt{-p} - \frac{b}{3a}$$
$$x_3 = -\sqrt{-p} - \frac{b}{3a}$$

To be more specific, it is analyzed based on the following conditions:

a. Case 2(a)

If $p > 0$, then $\Delta = \frac{q^2}{4} + \frac{p^3}{27} = \frac{p^3}{27} > 0$. Based on the case 2, the value of x_1 is a real number and x_2, x_3 are complex number. Thus, the characteristics of the roots are one real root and two conjugate complex roots.

b. Case 2(b)

If $p < 0$, then $\Delta = \frac{q^2}{4} + \frac{p^3}{27} = \frac{p^3}{27} < 0$. Based on the case 2, the value of x_1, x_2, and x_3 are real number. Thus, the characteristics of the roots are three distinct real roots.

3. Case 3:

If $p = 0$ and $q \neq 0$, then $\Delta = \frac{q^2}{4} > 0$. The equation $z^3 + pz + q = 0$, becomes
$z^3 + q = 0$
$\left(z + \sqrt[3]{q}\right)\left(z^2 - \sqrt[3]{q}z + \sqrt[3]{q^2}\right) = 0$
$z_1 = -\sqrt[3]{q}$ dan $z_{2,3} = \frac{-(-\sqrt[3]{q}) \pm \sqrt{(-\sqrt[3]{q})^2 - 4\left(\sqrt[3]{q^2}\right)}}{2} \Leftrightarrow z_{2,3} = \frac{\sqrt[3]{q}}{2}(1 \pm i\sqrt{3})$
Thus, $z_1 = -\sqrt[3]{q},\ z_2 = \frac{\sqrt[3]{q}}{2}(1 + i\sqrt{3})$ and $z_3 = \frac{\sqrt[3]{q}}{2}(1 - i\sqrt{3})$
Substitute z_1, z_2 dan z_3 into $x = z - t$, with $t = \frac{b}{3a}$.
$x_1 = -\sqrt[3]{q} - \frac{b}{3a}$
$x_2 = \frac{\sqrt[3]{q}}{2}(1 + i\sqrt{3}) - \frac{b}{3a}$
$x_3 = \frac{\sqrt[3]{q}}{2}(1 - i\sqrt{3}) - \frac{b}{3a}$
Thus, the characteristics of the roots are one real root and two conjugate complex roots.

4. Case 4:

If $p = 0$ and $q \neq 0$. Then, $\Delta = \frac{0^2}{4} + \frac{0^3}{27} = 0$. The equation $z^3 + pz + q = 0$, becomes
$z^3 + pz + q = 0$
$z^3 = 0 \Leftrightarrow z_{1,2,3} = 0$
Substitute z_1, z_2 and z_3 into $x = z - t$, with $t = \frac{b}{3a}$.
$x_{1,2,3} = -\frac{b}{3a}$

Thus, the characteristics of the roots are three equal real roots.
In the research results to form the Cardano formula, the graph and the influence that occurs on the graph are analyzed first. Then, the canonical equation is formed to direct the basic form in forming the Cardano formula. based on the canonical equation and the formation of the Cardano formula, the effect of the coefficient of the equation $z^3 + pz + q =$ and delta (Δ) can be analyzed, where eight characteristic roots are found. By using this, the roots can be found directly, but if the form of the equation has formed a canonical equation, if not then it is formed first to the canonical equation. The results of this study are a development of previous research, where previous research only discussed one topic such as only discussing the way to find the characteristics of a cubic equation that has real roots and the effect of the coefficient of the equation $z^3 + pz + q = 0$ in determining the roots.

4. Conclusion

Based on the results and discussion, every cubic equation $ax^3 + bx^2 + cx + d = 0$ can be reduced to a simple form namely the form $z^3 + pz + q = 0$, by supposing $x = z - t$, where $t = \frac{b}{3a}$ which means that the graph pattern has a root shift of t. The values obtained are,

$$p = -\frac{b^2}{3a^2} + \frac{c}{a}$$

and

$$q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}$$

From the canonical equation $z^3 + pz + q = 0$ will be formed Cardano's by supposing $z = w - t$. It is obtained that,

$$z_{1,2,3} = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3}$$

Characteristics of the roots of the cubic equation $z^3 + pz + q = 0$, are as follows:
(a) For $p \neq 0$ and $q \neq 0$. The roots of the equation are,
$x_1 = u + v - \frac{b}{3a}$
$x_2 = -\frac{1}{2}(u + v) + \frac{1}{2}i\sqrt{3}(u - v) - \frac{b}{3a}$
$x_3 = -\frac{1}{2}(u + v) - \frac{1}{2}i\sqrt{3}(u - v) - \frac{b}{3a}$

Divided into the following cases:
1. If $p > 0$ then $\Delta > 0$, there are one real root and two conjugate complex roots.
2. If $p < 0$ then $\Delta > 0$, there are one real root and two conjugate complex roots.
3. If $p < 0$ then $\Delta < 0$, there are three distinct real roots.
4. If $p < 0$ then $\Delta = 0$, there are three real roots with two equal real roots.

(b) For $p \neq 0$ and $q = 0$. The roots of the equation are,

$x_1 = -\frac{b}{3a}$

$x_2 = \sqrt{-p} - \frac{b}{3a}$

$x_3 = -\sqrt{-p} - \frac{b}{3a}$

Divided into the following cases:
1. If $p > 0$ then $\Delta > 0$, there are one real root and two conjugate complex roots.
2. If $p < 0$ then $\Delta < 0$, there are three distinct real roots.

(c) For $p \neq 0$ and $q = 0$ then $\Delta > 0$, there are one real root and two conjugate complex roots. The roots of the equation are,

$x_1 = -\sqrt[3]{q} - \frac{b}{3a}$

$x_2 = \frac{\sqrt[3]{q}}{2}(1 + i\sqrt{3}) - \frac{b}{3a}$

$x_3 = \frac{\sqrt[3]{q}}{2}(1 - i\sqrt{3}) - \frac{b}{3a}$

(d) For $p = 0$ and $q = 0$ then $\Delta = 0$, there are three equal real roots. The roots of the equation are,

$x_{1,2,3} = -\frac{b}{3a}$

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