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# Analysis of Nonlinear Oscillation Models with External Forcing Using the Multiple Scales Method

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**Abstract-** Nonlinear effects accompanied by external forces can cause the behaviour of the system to become more complex and difficult to explain using linear analysis. Therefore, analytical methods are needed to obtain approximate solutions. This paper presents an analysis of approximate solutions to nonlinear oscillation models subject to periodic external forces. The analysis was conducted using the Multiple Scales Method, a perturbation technique for obtaining asymptotic solutions to nonlinear differential equations. This approach is carried out by introducing several time scales and developing solutions as series in ε. The differential equations that model the system are analysed to orders. $O(1)$ and $O(\varepsilon)$ to obtain approximate solutions that describe the oscillation dynamics of the system. The analysis was performed under two main conditions: when the external force frequency approached the system's natural frequency (main resonance) and when the two were not close. In the non-resonance condition, several special cases were also examined: non-resonant, superharmonic resonance, subharmonic resonance, and low excitation frequency. The results show that first-order asymptotic solutions agree well with numerical solutions. The system response is influenced by parameters such as the amplitude and frequency of the external force, as well as the damping parameter. These findings support further research on more complex nonlinear systems and have practical applications in the design of vibration absorbers and rotating mechanical components to control resonance and improve system stability.

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## 1. Introduction

Nonlinear oscillation systems are an important topic in applied mathematics and system dynamics because they appear in many physical and engineering phenomena, such as mechanical systems, building structures, electrical circuits, and fluid dynamics. Unlike linear systems, nonlinear systems exhibit more complex behaviour because their responses are influenced by model parameters, which can cause phenomena such as resonance, bifurcation, and changes in solution stability (Nayfeh et al., 1979). These characteristics analyse nonlinear systems that are essential in practical applications, particularly for understanding and controlling vibration and dynamic responses in engineering systems.

Nonlinear oscillation system models are typically formulated as nonlinear differential equations that are difficult to solve analytically. Therefore, various approximation methods have been developed to obtain approximate solutions, including the perturbation method. This method utilises small parameters in the system to develop solutions in series, enabling systematic analysis of their characteristics (Nayfeh, 1981; Kevorkian et al., 1996).

One of the most widely used perturbation methods is the multiple scales method. This method was introduced to overcome the limitations of conventional perturbation methods, particularly in avoiding the emergence of secular terms that can cause solutions to become unbounded over time (Atiyah et al, 2024). By introducing multiple time scales, this method provides a more stable solution approximation and describes the system's dynamics over a longer period. (Bogolyubov et al., 1961; Vakakis et al., 1993). In practical contexts, this capability is particularly useful for analysing long-term system responses under periodic excitation, which is common in vibration-related engineering problems.

In nonlinear oscillation systems, resonance is a phenomenon often studied. Resonance occurs when the frequency of the external force approaches the system's natural frequency, causing an increase in the amplitude of the oscillation. Several types of resonance that are often discussed include primary, superharmonic, and subharmonic resonances (Shen et al., 2020). Previous studies have shown that nonlinear systems can exhibit complex dynamics around resonance conditions, including changes in solution stability, bifurcations, and chaotic behaviour (Li et al., 2007; Wang et al., 2020). In addition, several studies have shown that nonlinear oscillation systems can exhibit various complex dynamic phenomena, such as changes in solution stability, the emergence of bifurcations, and even chaotic behaviour when the system is near certain resonance conditions (Strogatz, 2015; Fritzkowski et al., 2021). Analysis of these phenomena is important because it can provide a deeper understanding of the dynamic characteristics of nonlinear systems. These findings provide the basis for analysing how system parameters influence dynamic responses and motivate the use of analytical approaches, such as the multiple-scales method, in the present study.

Based on this, this study aims to analyse approximate solutions to a nonlinear oscillation model subject to periodic external forces using the multiple-scale method. The analysis was conducted under several excitation conditions: primary resonance, superharmonic resonance, subharmonic resonance, and low-frequency excitation. The results of this study are expected to provide a better understanding of the dynamics of nonlinear oscillation systems. In particular, this study is expected to contribute to the prediction and control of resonance behaviour in practical applications.

## 2. Methods

This research employs an analytical approach to study the behaviour of a nonlinear oscillatory system subject to a periodic external force. The analysis is carried out using the multiple scales method, a perturbation technique widely used to obtain approximate solutions to nonlinear differential equations (Rafei, 2012). Compared to regular perturbation methods, the Multiple Scales Method is more effective in handling resonance and avoiding secular terms that may lead to unbounded solutions. By introducing multiple time scales, this method provides uniformly valid approximations. Through this method, the system dynamics can be examined by considering the gradual changes in amplitude and phase that occur over time.

### (a) Oscillation Model with External Force

Consider an oscillation model given by:

$$\ddot{u} + \omega_0^2 u = \varepsilon(-\mu\dot{u} - \alpha\ddot{u}^2) + F(t),\ u(0) = x_0, \dot{u}(0) = \dot{x}_0 \tag{2.1}$$

where $\varepsilon \ll 1$ represents weak nonlinearity and damping effects in the system. This assumption allows the solution to be expanded perturbatively and is commonly used in modelling weakly nonlinear oscillatory systems encountered in engineering applications. F(t) is a function representing the external force in the form of

$$F(t) = K\cos\Omega t = \frac{K}{2}e^{\Omega t} + c.c$$

The parameter $\mu > 0$represents the magnitude of damping in the system. The parameter $\alpha$ describes the stiffness characteristic of the spring; $\alpha > 0$ corresponds to a hard spring, while $\alpha < 0$ corresponds to a soft spring. The parameter $\omega_0 \neq 0$ is called the natural frequency, whereas $K$and $\Omega$ denote the amplitude and the frequency of the external force (excitation amplitude and excitation frequency), respectively. The notation c.c. represents the complex conjugate of the preceding term. The order of the amplitude $K \in \mathbb{R}$ and the frequency $\Omega \in \mathbb{R}$ is adjusted according to the problem under consideration.

### (b) Multiple Scales Method (MSM)

The multiple scales method is used to obtain an asymptotic solution. Assume that the solution can be expressed as

$$u(t) = u_0(T_0, T_1) + \varepsilon u_1(T_0, T_1) + \cdots$$

where $T_n = \varepsilon^n t$ (Nayfeh, 2008).

The introduction of multiple time scales $T_0$ = t and $T_1 = \varepsilon t$ allows the separation of fast and slow dynamics in the system. The variable $T_0$ represents the fast oscillatory motion, while $T_1$ captures the slow variation of amplitude and phase due to nonlinear and damping effects.
Note that

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon\left(\frac{\partial}{\partial T_1}\right) + \varepsilon^2\frac{\partial}{\partial T_2} + \cdots \tag{2.2}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2\left(2\frac{\partial^2}{\partial T_0 \partial T_2} + \frac{\partial^2}{\partial T_1^2}\right) + \cdots \tag{2.3}$$

Furthermore, we introduce the notation $D_0^2 = \frac{\partial^2}{\partial T_0^2}, D_0 D_1 = \frac{\partial^2}{\partial T_0 \partial T_1}$ and so on.
Substituting the multiple-scale expansion into the derivatives yields

$$\dot{u} = (D_0 u_0 + \varepsilon D_1 u_0) + \varepsilon(D_0 u_1 + \varepsilon D_1 u_1) + \cdots \tag{2.4}$$

$$\ddot{u} = (D_0^2 u_0 + 2\varepsilon D_0 D_1)u_0 + \varepsilon^2 D_1^2 u_0) + \varepsilon(D_0^2 u_1 + 2\varepsilon D_0 D_1 u_1 + \varepsilon^2 D_1^2 u_1) + \cdots \tag{2.5}$$

This transformation enables the governing equation to be analysed at different orders of ε, allowing the identification of terms that contribute to the leading-order behaviour and higher-order corrections.

## 3. Results and Discussion

The solution of equation (2.1) is analysed for two cases, namely when $\Omega \approx \omega_0$ and when $\Omega \neq \omega_0$.

### (a) Case I: $\Omega \approx \omega_0$

The case $\Omega \approx \omega_0$ is known as primary (or main) resonance (Wang et al, 2024). A detuning parameter $\sigma =$

$O(1)$is introduced and written as $\Omega = \omega_0 + \varepsilon\sigma$. For the excitation amplitude, we assume $K = O(\varepsilon)$, that is $K = \varepsilon k$, $w$here $k = O(1)$. Thus, equation (2.1) becomes

$$\ddot{u} + \omega_0^2 u = \varepsilon(-\mu\dot{u} - \alpha\ddot{u}^2) + \frac{\varepsilon k}{2} e^{i(\omega_0 + \varepsilon\sigma)t} + c.c \tag{3.1}$$

Using the multiple scales method, the governing equation is then examined at each order of $\varepsilon$, namely

$$O(1): D_0^2 u_0 + \omega_0^2 u_0 = 0 \tag{3.2}$$

$$O(\varepsilon): D_0^2 D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 - \mu D_0 u_0 - \alpha(D_0^2 u_0)^2 + \frac{k}{2} e^{i(\omega_0 T_0 + \sigma T_1)} + c.c \tag{3.3}$$

From equation (3.2), the solution is obtained as

$$u_0(T_0, T_1) = A(T_1) e^{i\omega_0 T_0} + \bar{A}(T_1) e^{-i\omega_0 T_0} \tag{3.4}$$

The $O(\varepsilon)$ equation can be written as

$$D_0^2 u_1 + \omega_0^2 u_1 = -\alpha\omega_0^4 A^2 e^{2i\omega_0 T_0} + \left[-2iw_0(D_1 A) - \mu\, i\omega_0 A + \frac{k}{2} e^{i\sigma T_1}\right] e^{i\omega_0 T_0} + c.c \tag{3.5}$$

In equation (3.5), a secular term appears, causing the asymptotic approximation to become non-uniform. The secular term can be removed from the particular solution of equation (3.5) by choosing A as the solution of

$$2iw_0(D_1 A) + \mu\, i\, \omega_0 A - \frac{k}{2} e^{i\sigma T_1} = 0 \tag{3.6}$$

Let $A(T_1) = \frac{1}{2} a e^{i\beta}$ where $a = a(T_1)$ and $\beta = \beta(T_1)$. Substituting this expression into equation (3.5) yields

$$i\omega_0[a' + ia\beta'] + i\frac{1}{2}\mu\omega_0 a = \frac{k}{2} e^{i\sigma T_1 - i\beta} \tag{3.7}$$

Next, by considering the real and imaginary parts of equation (3.7), we obtain

$$Re: a\beta' = -\frac{k}{2\omega_0}\cos(\sigma T_1 - \beta) \tag{3.8}$$

$$Im: a' = -\frac{1}{2}\mu a + \frac{k}{2\omega_0}\sin(\sigma T_1 - \beta) \tag{3.9}$$

Note that equations (3.7) and (3.8) form a non-autonomous system. Define $\gamma = \sigma T_1 - \beta \Rightarrow \gamma' = \sigma - \beta'$, thus the system becomes

$$\begin{aligned} a' &= -\frac{1}{2}\mu a + \frac{1}{2}\frac{k}{\omega_0}\sin(\gamma) \\ \gamma' &= \sigma + \frac{1}{2}\frac{k}{\omega_0 a}\cos(\gamma) \end{aligned} \tag{3.10}$$

Therefore, equations (3.9) form an autonomous system.

## 1. Steady-State Solution

The steady-state solution of the autonomous system (3.10) is obtained when $a' = 0 = \gamma'$

$$\begin{aligned} \mu a_s &= \frac{k}{\omega_0}\sin\gamma_s \\ -2\sigma a_s &= \frac{k}{\omega_0}\cos(\gamma_s) \end{aligned} \tag{3.11}$$

The point $(a_{s'}\gamma_s)$that satisfies system (3.11) is called a singular point. By squaring both equations and adding them, we obtain

$$\sigma = \pm\sqrt{\frac{k^2}{4w_0^2 a_s^2} - \frac{\mu^2}{4}} \tag{3.12}$$

Equation (3.12) represents an implicit equation for the frequency response, where the steady-state amplitude $a_s$ is expressed as a function of the detuning parameter $\sigma$ and the excitation amplitude $k$. This equation is referred to as the frequency-response equation. The steady-state solution corresponding to equation (3.12) is given by

$$u_s = a_s\cos(\Omega t - \gamma_s) + O(\varepsilon) \tag{3.13}$$

## 2. Stability of the Steady-State Solution

In this section, the stability of the steady-state solution is examined at the singular point. $(a_s, \gamma_s)$ is perturbed. Let

$$a = a_s + a_1 \tag{3.14}$$

$$\gamma = \gamma_s + \gamma_1 \tag{3.15}$$

where $a_1$and $\gamma_1$ represent small perturbation terms. Note that $a_s$and $\gamma_s$satisfy equations (3.11)

Substituting equations (3.14) and (3.15) into the differential equations governing the frequency response and linearising the terms containing $a_1$and $\gamma_1$, we obtain the following system of differential equations

$$\begin{aligned} a_1' &= -\frac{1}{2}\mu a_1 + \frac{1}{2}\frac{k}{\omega_0}\gamma_1\cos(\gamma_s) \\ \gamma_1' &= -\frac{1}{2}\frac{k\, a_1\cos(\gamma_s) + k a_s(\gamma_1)\sin(\gamma_s)}{\omega_0 a_s^2} \end{aligned} \tag{3.16}$$

The linear stability is determined from the sign of the eigenvalues of the Jacobian matrix. Based on equation (3.11), the Jacobian matrix is

$$J = \begin{pmatrix} -\frac{1}{2}\mu & -\sigma a_s \\ \frac{\sigma}{a_s} & -\frac{1}{2}\mu \end{pmatrix} \tag{3.17}$$

The characteristic equation is given by

$$P(\lambda) = \lambda^2 - (trace\ J)\lambda + \det J = \lambda^2 + \mu\lambda + \left(\frac{1}{4}\mu^2 + \sigma^2\right) \tag{3.18}$$

Thus, the eigenvalues are

$$\lambda_{1,2} = -\frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4\left(\frac{1}{4}\mu^2 + \sigma^2\right)} = -\frac{\mu}{2} \pm \frac{1}{2}i\sigma \tag{3.19}$$

Since the real parts of both eigenvalues are negative, the steady-state solution is stable under perturbations.

## 3. Plot of The Asymtotic and Numerical Solutions for the Primary Resonance Case

In this section, the asymptotic solution obtained using the Multiple Scales Method (MSM) is compared with the numerical solution. The parameters are chosen as $\varepsilon = 0.01, \alpha = 5, \mu = 0.5, \sigma = 1, \omega_0 = 1, k = 1$ .

The initial conditions are given by

$$u(0) = 0.1, u'(0) = 0, a(0) = 0.1, \gamma(0) = 0$$

Figure 1 confirms that the first-order asymptotic solution obtained by the Multiple Scales Method provides a good approximation to the numerical solution in the primary resonance case.

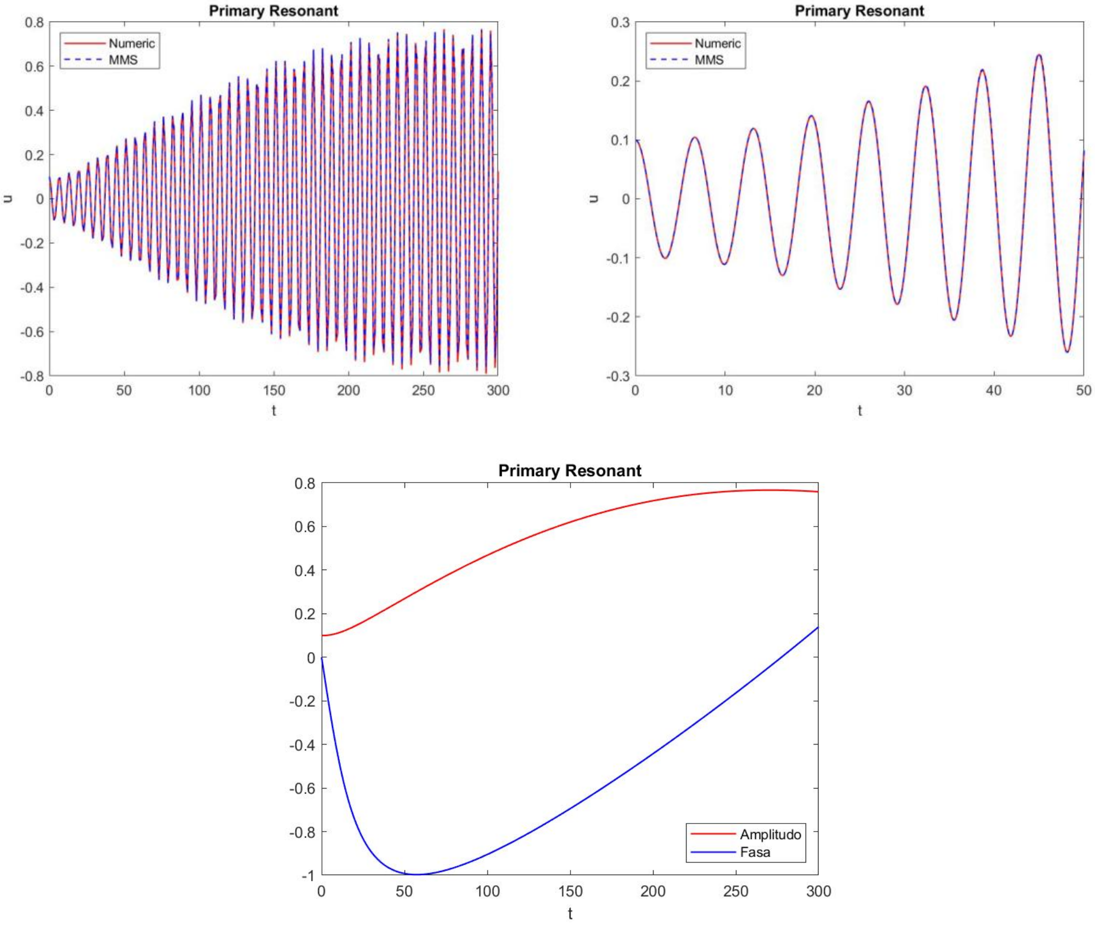

**Figure 1.** Primary resonance case

### (b) Case II: $\Omega \neq \omega_0$

When $\Omega$ is far from $\omega_0$The effect of the external excitation becomes very small. To observe the contribution of the excitation amplitude, we assume $K = O(1)$, so that $F(t) = K\cos(\Omega T_0)$. Thus, the governing equation for the oscillation model becomes

$$\ddot{u} + \omega_0^2 u = \varepsilon(-\mu\dot{u} - \alpha\ddot{u}^2) + \frac{K}{2}e^{i\Omega T_0} + c.c \tag{3.20}$$

Using a procedure analogous to Case I, the equation $O(1)$ is

$$D_0^2 u_0 + \omega_0^2 u_0 = K\cos(\Omega T_0) = \frac{K}{2}e^{i\Omega T_0} + c.c \tag{3.21}$$

The $O(1)$ The solution is given by

$$u_0 = A(T_1)e^{i\omega_0 T_0} + \Lambda e^{i\Omega T_0} + c.c \tag{3.22}$$

Where $\Lambda = \frac{K}{2(\omega_0^2 - \Omega^2)}$.

For the $O(\varepsilon)$ terms, the governing differential equation is

$$D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 - \mu D_0 u_0 - \alpha(D_0^2 u_0)^2 \tag{3.23}$$

Substituting equation (3.22) into equation (3.23), we obtain

$$D_0^2u_1+\omega_0^2u_1=G_1e^{i\omega_0T_0}+G_2e^{i\Omega T_0}+G_3e^{2i\omega_0T_0}+G_4e^{2i\Omega T_0}+G_5e^{iT_0(\omega_0+\Omega)} + G_6e^{iT_0(\Omega-\omega_0)}+ (\omega_0^4A\bar{A}+2\Lambda^2\Omega^4)+c.c \tag{3.24}$$

Where
$G_1=-2i\omega_0(D_1A)-\mu i\omega_0A,\quad G_2=-\mu i\Omega\Lambda,\quad G_3=-\alpha(\omega_0^4A^2),\ G_4=-\alpha\Lambda^2\Omega^4,\ G_5=-2\alpha\omega_0^2A\Lambda\Omega^2,$
$G_6=-2\alpha\omega_0^2\bar{A}\Lambda\Omega^2$

In addition to the term containing $e^{i\omega_0T_0}$, secular (or near-secular) terms may arise when $\Omega\approx 0\ (O(\varepsilon))$(see $G_5$) or when $m\omega_0+n\Omega\approx\omega_0$, with $|m|+|n|=2$. The latter case is referred to as secondary resonance.

Therefore, when applying the solvability condition (eliminating the secular terms), several cases must be considered (El-Dib, 2023):

- Non-resonant case: $\Omega$ is sufficiently far from $0$, $\omega_0/2$, or $2\omega_0$.
- Superharmonic resonance: $2\Omega\approx\omega_0$.
- Subharmonic resonance: $\Omega/2\approx\omega_0$.
- Low-frequency excitation: $\Omega\approx 0$.

## 1. Nonresonant Case

In the nonresonant case, the solvability condition gives $G_1=0$, which yields

$$-2i\omega_0(D_1A)-\mu i\omega_0A=0 \tag{3.25}$$

Substituting $A(T_1)=\frac{1}{2}ae^{i\beta}$, where $a=a(T_1)$and $\beta=\beta(T_1)$, gives

$$u_0=a(T_1)\cos\big(\omega_0t+\beta(T_1)\big)+\frac{K}{\omega_0^2-\Omega^2}\cos\Omega t+O(\varepsilon) \tag{3.29}$$

where $a(T_1)=a_0e^{-\frac{1}{2}\mu t}, a=a(T_1), b=b(T_1)$ which can be determined from the initial conditions.

Next, the plot of the nonresonant solution (MSM and numerical) is presented using the specified parameters.

$$\varepsilon=0.01,\alpha=5,\mu=0.5,\sigma=1,\omega_0=1,K=1,\Omega=10$$

Initial conditions:

$$u(0)=1+2\Lambda,u'(0)=0,\ a(0)=1,\beta(0)=0$$

Figure 2 shows the comparison between the numerical solution and the first-order asymptotic approximation for the nonresonant case, where $\Omega$ is sufficiently far from $0$, $\omega_0/2$, or $2\omega_0$. The graphs of both solutions overlap, indicating that the first-order approximation agrees well with the numerical solution. In this case, the free oscillation (homogeneous solution) decays.

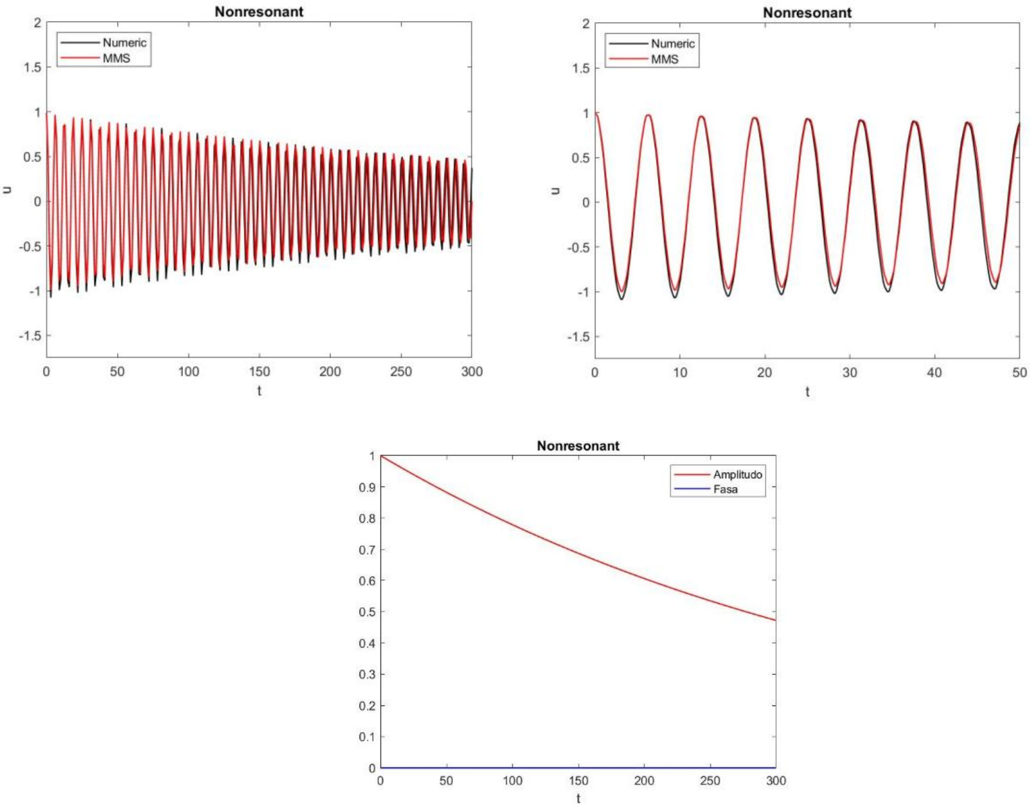

**Figure 2.** Nonresonant Case

## 2. Superharmonic Resonance ($2\Omega \approx \boldsymbol{\omega_0}$)

In the nonresonant case, the solvability condition gives $G_1 = 0$, which yields

$$-i\omega_0 a' + \omega_0 a\beta' - \frac{1}{2} i\mu\omega_0 a - \alpha\Lambda^2\Omega^4(\cos(\sigma T_1 - \beta) + i\sin(\sigma T_1 - \beta)) = 0 \tag{3.30}$$

Taking the real and imaginary parts yields

$$a' = -\frac{1}{2}\mu a - \frac{\alpha}{\omega_0}\Lambda^2\Omega^4 \sin(\sigma T_1 - \beta) \tag{3.31}$$

$$\beta' = \frac{\alpha\Lambda^2\Omega^4}{\omega_0 a}\cos(\sigma T_1 - \beta) \tag{3.32}$$

Let $\gamma = \sigma T_1 - \beta$ so that $\beta' = \sigma - \gamma'$, Substituting this into the above equations gives

$$a' = -\frac{1}{2}\mu a - \frac{\alpha}{\omega_0}\Lambda^2\Omega^4 \sin(\gamma) \tag{3.33}$$

$$\gamma' = \sigma - \frac{\alpha\Lambda^2\Omega^4}{\omega_0 a}\cos(\gamma) \tag{3.34}$$

Since $2\Omega = \omega_0 + \varepsilon\sigma$ and $\gamma = \sigma T_1 - \beta$, the solution of equation (3.20) is

$$u = a(T_1)\cos(2\Omega \mathrm{t} - \gamma) + \frac{K}{(\omega_0^2 - \Omega^2)}\cos e^{i\Omega t} + \mathcal{O}(\varepsilon) \tag{3.35}$$

Next, the plot of the nonresonant solution (MSM and numerical solution) is presented using the parameters

$$\varepsilon = 0.01, \alpha = 5, \mu = 0.5, \sigma = 1, \omega_0 = 1, K = 1$$

Initial conditions:

$$u(0) = 1 + 2\Lambda, u'(0) = 0,\ a(0) = 1, \beta(0) = 0$$

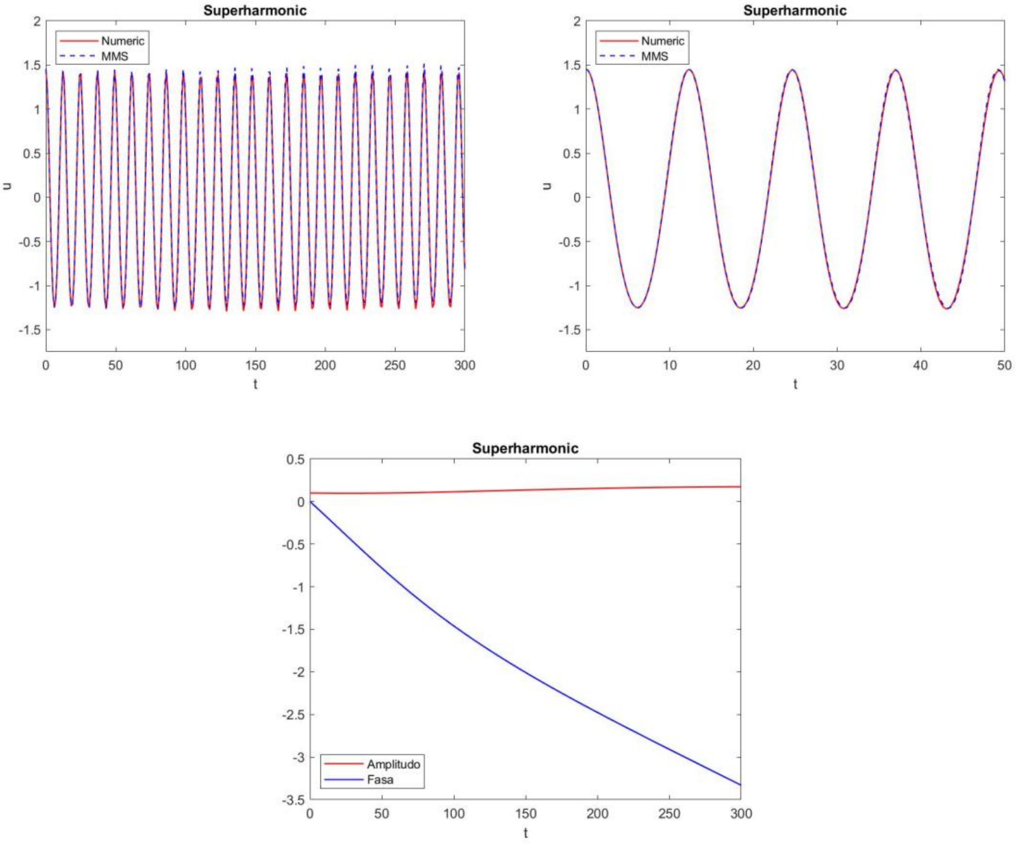

**Figure 3.** Superharmonic resonance case

Figure 3 compares the numerical solution with the first-order asymptotic approximation for the superharmonic resonance case. The two curves overlap, indicating that the asymptotic approximation provides a good representation of the numerical solution. In this case, the free oscillation does not decay to zero.

## 3. Subharmonic Resonance ($\Omega/2 \approx \omega_0$)

The detuning parameter is written as $\Omega = 2\omega_0 + \epsilon\sigma$ . The solvability condition gives $G_1 + G_6 e^{i\sigma T_1} = 0$, which is given by

$$i\omega_0 a' - \omega_0 \beta' a + \frac{1}{2} i\mu\omega_0 a + \alpha\omega_0^2 a\Lambda\Omega^2(\cos(\sigma T_1 - 2\beta) + i\sin(\sigma T_1 - 2\beta)) = 0 \tag{3.36}$$

Taking the real and imaginary parts yields

$$a' = -\frac{1}{2}\mu a - \alpha\omega_0 a\Lambda\Omega^2 \sin(\sigma T_1 - 2\beta) \tag{3.37}$$

$$\beta' = \alpha\omega_0\Lambda\Omega^2 \cos(\sigma T_1 - 2\beta) \tag{3.38}$$

Let $\gamma = \sigma T_1 - 2\beta$ so that $\beta' = \frac{\sigma}{2} - \frac{1}{2}\gamma'$, thus

$$a' = -\frac{1}{2}\mu a - \alpha\omega_0 a\Lambda\Omega^2 \sin(\gamma) \tag{3.39}$$

$$\gamma' = \sigma - 2\alpha\omega_0\Lambda\Omega^2 \cos(\gamma) \tag{3.40}$$

The solution of equation (3.20) becomes

$$u = a(T_1)\cos 2(\Omega t - \gamma) + \frac{K}{(\omega_0^2 - \Omega^2)}\cos(\Omega t) + O(\varepsilon) \tag{3.41}$$

Next, the plot of the subharmonic resonance case (MSM and numerical solution) is presented using the parameters $\varepsilon = 0.01, \alpha = 5, \mu = 0.5, \sigma = 1, \omega_0 = 1, K = 1$

Initial conditions:

$$u(0) = 1 + 2\Lambda, u'(0) = 0,\ a(0) = 1, \beta(0) = 0$$

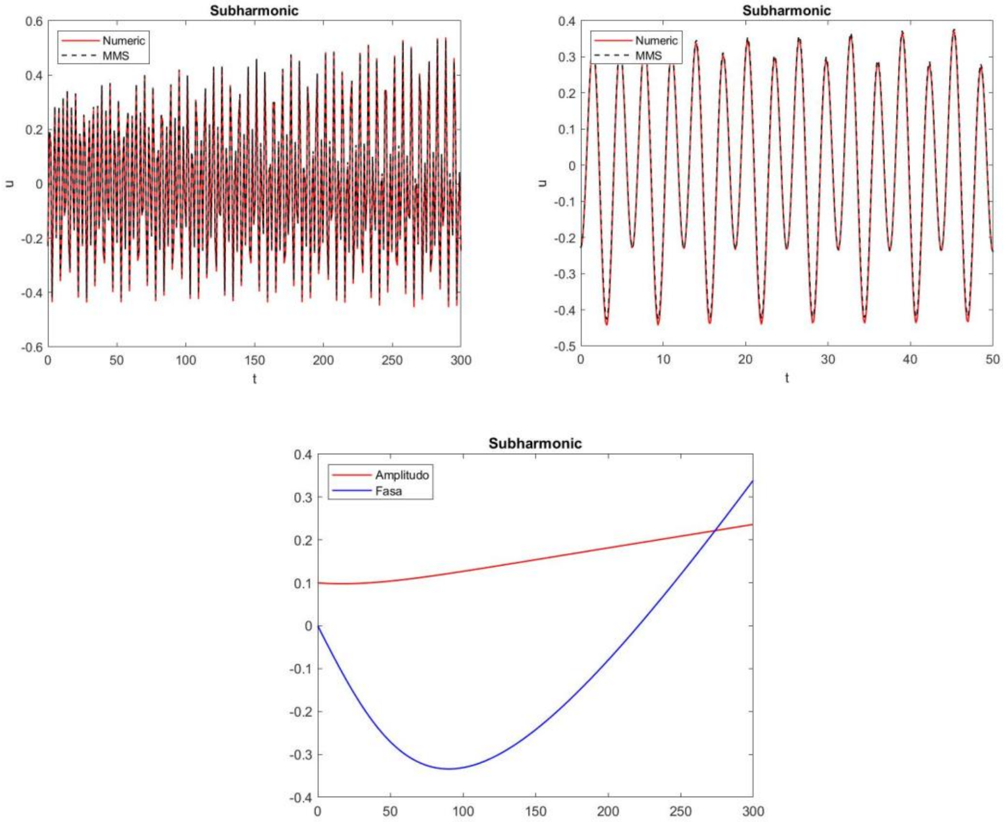

**Figure 4.** Subharmonic resonance case

## 4. Low-Frequency Excitation ($\Omega \approx 0$)

The detuning parameter is written as $\Omega = \varepsilon\sigma$. The solvability condition gives $G_1 + G_5 e^{i\sigma T_1} = 0$ which yields

$$i\omega_0 a' - \omega_0 a\beta' + \frac{1}{2}\mu i\omega_0 a + \alpha\omega_0^2 a\Lambda\Omega^2(\cos(\sigma T_1) + i\sin(\sigma T_1)) = 0 \tag{3.42}$$

Taking the real and imaginary parts gives

$$a' = -\frac{1}{2}\mu a - \alpha\omega_0 a\Lambda\Omega^2\sin(\gamma) \tag{3.43}$$

$$\gamma' = \sigma - 2\alpha\omega_0\Lambda\Omega^2\cos(\gamma) \tag{3.44}$$

Solving (3.44) – (3.45) yields

$$a(T_1) = C_1\,\mathrm{e}^{-\frac{1}{2}\mu T_1 + \frac{1}{\sigma}\alpha\omega_0\Lambda\Omega^2\cos(\sigma T_1)} \tag{3.45}$$

$$\beta(T_1) = \frac{1}{\sigma}\alpha\omega_0\Lambda\Omega^2\sin(\sigma T_1) + C_2 \tag{3.46}$$

Remember that $\Omega = \varepsilon\sigma$, the solution becomes

$$u = a(T_1)\cos(\beta + \omega_0 t) + \frac{K}{(\omega_0^2 - \Omega^2)}\cos \Omega t \tag{3.47}$$

Next, the plot of the Low-Frequency Excitation (MSM and numerical solution) is presented using the parameters

$$\varepsilon = 0.01, \alpha = 5, \mu = 0.5, \sigma = 1, \omega_0 = 1, K = 1$$

Initial conditions:

$$u(0) = 1 + 2\Lambda, u'(0) = 0,\ a(0) = 1, \beta(0) = 0$$

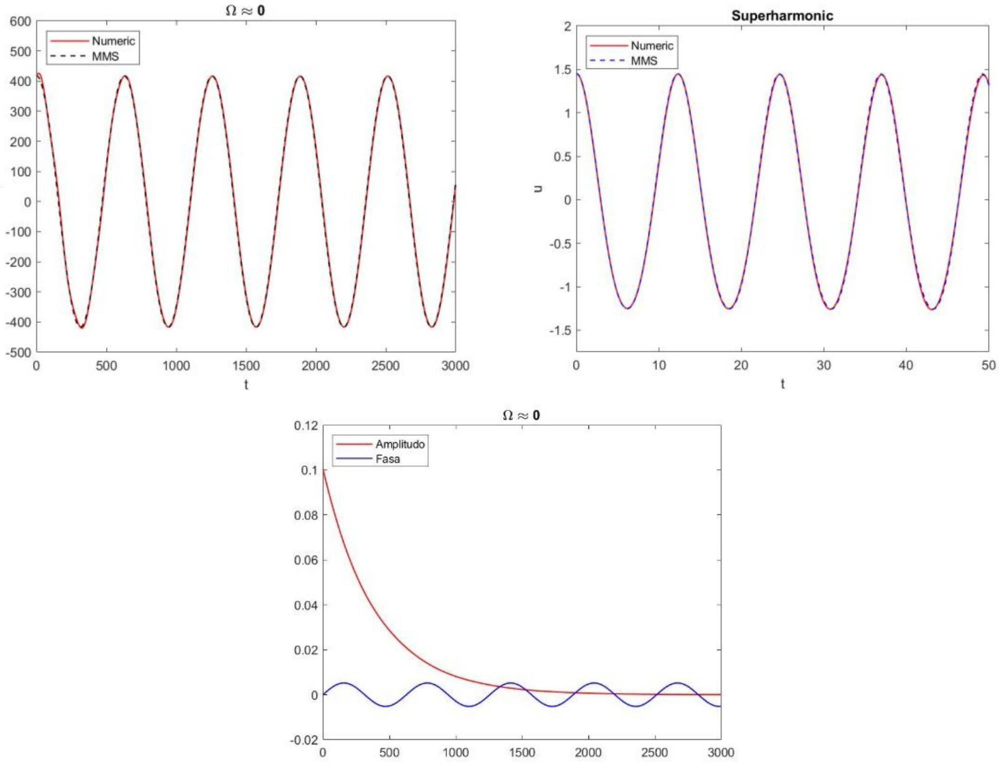

**Figure 5.** Low-frequency excitation case

Figure 5 shows the comparison between the numerical solution and the first-order asymptotic approximation when $\Omega \approx 0$. The two curves overlap, indicating that the first-order approximation provides a good representation of the numerical solution.

The results also indicate that the system response near resonance is highly sensitive to changes in excitation parameters. This behaviour is relevant in practical applications involving mechanical vibration systems, where controlling oscillation amplitude is essential to prevent excessive vibration. Therefore, the findings of this study can provide useful insights into the understanding and prediction of resonance behaviour in engineering systems.

Although the Multiple Scales Method provides satisfactory results for the system considered in this study, its application to more complex systems may present several challenges. The complexity of the analysis tends to increase with higher levels of nonlinearity or system dimensionality, making the analytical process more involved. In addition, the method relies on certain assumptions that may not always be fully satisfied in more general cases. Therefore, further development or integration with other approaches may be considered to obtain more accurate results for more complex systems.

## 4. Conclusion

Based on the results of this study, it can be concluded that the Multiple Scales Method can be effectively applied to analyse nonlinear oscillation models with external forcing to obtain approximate or asymptotic solutions. The method introduces multiple time scales, allowing the influence of small parameters to be analysed through a perturbation approach. The governing differential equation is expanded in terms of the small parameter $\varepsilon$, and solutions are obtained at orders $O(1)$and $O(\varepsilon)$, resulting in approximate solutions that describe the oscillatory behaviour of the system. By considering the external forcing as a time-dependent function, the system's dynamic response is shown to depend on several parameters, including the amplitude and frequency of the external force, as well as the system parameters. The results demonstrate that the first-order asymptotic solution provides a good approximation to the numerical solution and captures several dynamical behaviours of the system, including resonant and nonresonant cases.

These findings indicate that the proposed approach can be useful for understanding and predicting resonance phenomena in oscillatory systems, particularly regarding how system parameters influence vibration amplitude. This is relevant for applications involving vibration control and the design of mechanical systems where resonance needs to be avoided or reduced. In addition, further research may consider extending the present analysis to more complex systems, such as those involving coupled oscillations or higher-dimensional models, to broaden the method's applicability.

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